Compact IMU Kinematic Model with Extended Poses: Discretization, Jacobians, and Preintegration.

Charles Champagne Cossette
Department of Mechanical Engineering, McGill University
817 Sherbrooke Street West, Montreal QC H3A 0C3

November 23, 2022

Summary

It is possible to write the continuous-time IMU kinematics in the form

$$\dot{\mathbf{T}} = \mathbf{G}\mathbf{T} + \mathbf{T}\mathbf{U}, \tag{1}$$

which is a matrix-valued ODE linear in $\mathbf{T} \in SE_2(3)$. Assuming that $\mathbf{G}, \mathbf{U}$ are constant, this ODE can be integrated exactly to produce the following discretization scheme,

$$\mathbf{T}_k = \exp(\Delta t \mathbf{G}) \mathbf{T} \exp(\Delta t \mathbf{U}) \tag{2}$$

$$\triangleq \mathbf{G}_{k-1} \mathbf{T}_{k-1} \mathbf{U}_{k-1}. \tag{3}$$

Not only is this extremely compact and tractable, but we have also gotten rid of a typical assumption: that the IMU attitude is constant over the integration interval Δt. Left- and right-Jacobian derivations also become greatly simplified, and the left-Jacobian is fully independent of the state and input, including biases.

The biggest gain occurs during preintegration. Pages of algebra are replaced with a single step, obtained by directly iterating (3)

$$\mathbf{T}_j = \left(\prod_{k=i}^{j-1} \mathbf{G}_k \right) \mathbf{T}_i \left(\prod_{k=i}^{j-1} \mathbf{U}_k \right), \tag{4}$$

and the result is far more tractable than [1].

Contents

1 Introduction

The typical roboticist is familiar with the somewhat cumbersome, nonlinear kinematic equations that relate IMU measurements to the rate of change of a body's position, velocity, and attitude,

$$\dot{\mathbf{C}}_{ab}(t) = \mathbf{C}_{ab}(t)(\boldsymbol{\omega}_b(t))^\wedge, \tag{5}$$
$$\dot{\mathbf{v}}_a^{zw/a}(t) = \mathbf{C}_{ab}(t)\mathbf{a}_b(t) + \mathbf{g}_a, \tag{6}$$
$$\dot{\mathbf{r}}_a^{zw}(t) = \mathbf{v}_a^{zw/a}(t), \tag{7}$$

where $\boldsymbol{\omega}_b, \mathbf{a}_b$ denote the gyroscope and accelerometer measurements, respectively, without any bias or noise. These equations must be discretized for any practical use. Lightening the notation, the standard discretization scheme used throughout the literature [1] is the following[1]:

$$\mathbf{C}_k = \mathbf{C}_{k-1}\exp(\Delta t\boldsymbol{\omega}_{k-1}^\wedge), \tag{8}$$
$$\mathbf{v}_k = \mathbf{v}_{k-1} + \Delta t\mathbf{g} + \Delta t\mathbf{C}_{k-1}\mathbf{a}_{k-1}, \tag{9}$$
$$\mathbf{r}_k = \mathbf{r}_{k-1} + \Delta t\mathbf{v}_{k-1} + \frac{\Delta t^2}{2}(\mathbf{g} + \mathbf{C}_{k-1}\mathbf{a}_{k-1}). \tag{10}$$

Such a discretization involves two assumptions: 1) that the body-frame acceleration and angular velocity are held constant over the integration interval Δt and 2) that the body attitude $\mathbf{C}(t)$ is also constant over this interval. For IMUs with high rates relative to their motion profile, these assumptions have been shown to be appropriate.

Extended poses of the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{v} & \mathbf{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SE_2(3) \tag{11}$$

have often been used to represent the collective attitude-position-velocity state. Writing a continuous-time process model of the form $\dot{\mathbf{T}} = \mathbf{F}(\mathbf{T}, \mathbf{u})$, has typically been done in a non-compact component form, effectively implementing equations (8) - (10). This is not only cumbersome, but also requires the Jacobians, as well as preintegration, to be computed in a non-compact form. The work in [2, 3] achieves something close with a discrete-time process model of the form

$$\mathbf{T}_k = \boldsymbol{\Gamma}_{k-1}\phi(\mathbf{T}_{k-1})\boldsymbol{\Upsilon}_{k-1} \tag{12}$$

however, $\phi(\mathbf{T})$ is still a function that operates on the internal components of $\mathbf{T}$. This document will present the IMU process model in a slightly different, simpler way, but for the large part achieving the same benefits as [2, 3].

2 Compact Form

The continuous time IMU kinematics can be written as

$$\dot{\mathbf{T}} = \mathbf{G}\mathbf{T} + \mathbf{T}\mathbf{U} \tag{13}$$
$$\begin{bmatrix} \mathbf{C}\boldsymbol{\omega}^\wedge & \mathbf{C}\mathbf{a} + \mathbf{g} & \mathbf{v} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{g} & \mathbf{0} \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{v} & \mathbf{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{v} & \mathbf{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}^\wedge & \mathbf{a} & \mathbf{0} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

The matrix $\mathbf{G}$ is always constant. The matrix $\mathbf{U}$ is only a function of the IMU measurements. Assuming that $\mathbf{U}$ is constant between a small integration interval Δt between IMU measurements, the solution to (13) with initial condition $\mathbf{T}_{k-1}$ is

$$\mathbf{T}_k = \exp(\Delta t\mathbf{G})\mathbf{T}_{k-1}\exp(\Delta t\mathbf{U}) \tag{15}$$
$$\triangleq \mathbf{G}_{k-1}\mathbf{T}_{k-1}\mathbf{U}_{k-1}. \tag{16}$$

[1] The subscripts and superscripts have been omitted for readability.

The matrices $\mathbf{G}_{k-1}, \mathbf{U}_{k-1}$ can be computed in closed form by a direct series expansion of the matrix exponential

$$\mathbf{G}_{k-1} = \begin{bmatrix} \mathbf{1} & \Delta t\mathbf{g} & -(\Delta t^2/2)\mathbf{g} \\ 0 & 1 & -\Delta t \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$\mathbf{U}_{k-1} = \begin{bmatrix} \exp(\Delta t\boldsymbol{\omega}^\wedge) & \Delta t\mathbf{J}_\ell(\Delta t\boldsymbol{\omega})\mathbf{a} & (\Delta t^2/2)\mathbf{N}(\Delta t\boldsymbol{\omega})\mathbf{a} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

where an expression for matrix $\mathbf{N}(\cdot)$ is given in the appendix.

3 Process model Jacobians

3.1 With respect to the state

3.1.1 Right

The right Lie Jacobian of (16), with respect to the state, can be obtained by substiting perturbation definitions

$$\bar{\mathbf{T}}_k \exp(\delta\boldsymbol{\xi}_k^\wedge) = \mathbf{G}_{k-1}\bar{\mathbf{T}}_{k-1}\exp(\delta\boldsymbol{\xi}_{k-1}^\wedge)\mathbf{U}_{k-1} \tag{19}$$

$$\exp(\delta\boldsymbol{\xi}_k^\wedge) = \bar{\mathbf{T}}_k^{-1}\mathbf{G}_{k-1}\bar{\mathbf{T}}_{k-1}\underbrace{\mathbf{U}_{k-1}\mathbf{U}_{k-1}^{-1}}_{\mathbf{1}}\exp(\delta\boldsymbol{\xi}_{k-1}^\wedge)\mathbf{U}_{k-1} \tag{20}$$

$$\exp(\delta\boldsymbol{\xi}_k^\wedge) = \mathbf{U}_{k-1}^{-1}\exp(\delta\boldsymbol{\xi}_{k-1}^\wedge)\mathbf{U}_{k-1} \tag{21}$$

$$\delta\boldsymbol{\xi}_k = \mathbf{Ad}(\mathbf{U}_{k-1}^{-1})\delta\boldsymbol{\xi}_{k-1}. \tag{22}$$

3.1.2 Left

Following an identical procedure, but perturbing on the left will yield the left Lie Jacobian, and one obtains

$$\delta\boldsymbol{\xi}_k = \mathbf{Ad}(\mathbf{G}_{k-1})\delta\boldsymbol{\xi}_{k-1}, \tag{23}$$

which is **constant, and independent of the state and input**.

3.2 With respect to the input

Deriving the Jacobian with respect to the input is convenient, since both the noise and bias are simple additive quantities to this input, the Jacobians with respect to both the noise and bias can be easily obtained once the Jacobian of the input is known. However, this procedure is slightly more complicated than deriving the Jacobian with respect to the state, since $\mathbf{U}_{k-1}$ is not quite an element of $SE_2(3)$. However, $\mathbf{U}_{k-1}$ can be written as

$$\mathbf{U}_{k-1} = \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\Delta}_{k-1}} \underbrace{\begin{bmatrix} \exp(\Delta t\boldsymbol{\omega}^\wedge) & \Delta t\mathbf{J}_\ell(\Delta t\boldsymbol{\omega})\mathbf{a} & (\Delta t^2/2)\mathbf{N}(\Delta t\boldsymbol{\omega})\mathbf{a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{\mathbf{U}}_{k-1}} \tag{24}$$

where the matrix $\boldsymbol{\Delta}_{k-1}$ is referred to as a "time machine" by [4], and $\tilde{\mathbf{U}}_{k-1}$ now belongs to $SE_2(3)$. Defining the input to be $\mathbf{u} = [\boldsymbol{\omega}^\mathsf{T}\ \mathbf{a}^\mathsf{T}]^\mathsf{T}$ the matrix $\tilde{\mathbf{U}}_{k-1}$ can be written as

$$\tilde{\mathbf{U}}_{k-1} = \exp\left(\boldsymbol{\upsilon}(\mathbf{u}_{k-1})^\wedge\right), \qquad \boldsymbol{\upsilon}(\mathbf{u}) = \begin{bmatrix} \Delta t\boldsymbol{\omega} \\ \Delta t\mathbf{a} \\ (\Delta t^2/2)\mathbf{J}(\Delta t\boldsymbol{\omega})_\ell^{-1}\mathbf{N}(\Delta t\boldsymbol{\omega})\mathbf{a} \end{bmatrix}. \tag{25}$$

Following [4], it can be shown that $\boldsymbol{\upsilon}$ can be approximated as

$$\boldsymbol{\upsilon}(\mathbf{u}) \approx \bar{\boldsymbol{\upsilon}} + \underbrace{\begin{bmatrix} \Delta t & \mathbf{0} \\ \mathbf{0} & \Delta t \\ \Delta t^3(\frac{1}{12}\mathbf{a}^\wedge - \frac{1}{720}\Delta t^2 \mathbf{W}) & \frac{1}{2}\Delta t^2 \mathbf{J}_\ell^{-1}\mathbf{N} \end{bmatrix}}_{\boldsymbol{\Upsilon}} \underbrace{\begin{bmatrix} \delta\boldsymbol{\omega} \\ \delta\mathbf{a} \end{bmatrix}}_{\delta\mathbf{u}}, \tag{26}$$

$$\mathbf{W} = \boldsymbol{\omega}^{\wedge^2}\mathbf{a} + \boldsymbol{\omega}^\wedge(\boldsymbol{\omega}^\wedge\mathbf{a})^\wedge + (\boldsymbol{\omega}^{\wedge^2}\mathbf{a})^\wedge, \tag{27}$$

where it should be noted that this is an approximation depending on the assumption of a small Δt, and that $\boldsymbol{\Upsilon}$ is therefore not an exact Jacobian of $\boldsymbol{\upsilon}$, but rather an approximation. Nevertheless, approximating $\boldsymbol{\upsilon}$ this way leads to

$$\tilde{\mathbf{U}}_{k-1} = \exp(\boldsymbol{\upsilon}^\wedge) \approx \exp(\bar{\boldsymbol{\upsilon}}^\wedge)\exp((\boldsymbol{\mathcal{J}}(-\bar{\boldsymbol{\upsilon}})\boldsymbol{\Upsilon}\delta\mathbf{u})^\wedge) \tag{28}$$

$$\tag{29}$$

where $\boldsymbol{\mathcal{J}}(\cdot)$ is the left-Jacobian of $SE_2(3)$.

3.2.1 Right

Let $\mathbf{w} = [\mathbf{w}^{g^\mathsf{T}} \;\; \mathbf{w}^{a^\mathsf{T}}]^\mathsf{T}$ represent the IMU measurement noise. Since this quantity is simply additive to the IMU measurements, $\delta\mathbf{u} = \delta\mathbf{w}$. The right Lie jacobian is obtained by perturbing the process model on the right, as well as the noise about a zero nominal value $\mathbf{w} = \mathbf{0} + \delta\mathbf{w}$. This leads to

$$\bar{\mathbf{T}}_k \exp(\delta\boldsymbol{\xi}_k) \approx \underbrace{\mathbf{G}_{k-1}\mathbf{T}_{k-1}\boldsymbol{\Delta}_{k-1}\exp(\bar{\boldsymbol{\upsilon}}_{k-1}^\wedge)}_{\bar{\mathbf{T}}_k}\exp((\boldsymbol{\mathcal{J}}(-\bar{\boldsymbol{\upsilon}}_{k-1})\boldsymbol{\Upsilon}_{k-1}\delta\mathbf{w}_{k-1})^\wedge) \tag{30}$$

$$\delta\boldsymbol{\xi}_k \approx \underbrace{\boldsymbol{\mathcal{J}}(-\bar{\boldsymbol{\upsilon}}_{k-1})\boldsymbol{\Upsilon}_{k-1}}_{\mathbf{L}_{k-1}}\delta\mathbf{w}_{k-1}, \tag{31}$$

where $\mathbf{L}_{k-1}$ is the right Lie Jacobian of the process model with respect to the input.

3.2.2 Left

Following an identical procedure, but perturbing on the left will yield the left Lie Jacobian, and one obtains

$$\delta\boldsymbol{\xi}_k \approx \mathbf{Ad}(\bar{\mathbf{T}}_k)\mathbf{L}_{k-1}\delta\mathbf{w}_{k-1}. \tag{32}$$

3.3 Incorporating Bias

It is common to model the presence of a subtractive bias on the IMU measurements such that

$$\mathbf{u} = \bar{\mathbf{u}} - \mathbf{b} + \mathbf{w}^{\mathrm{IMU}} \tag{33}$$

where $\bar{\mathbf{u}}$ represents the "nominal" IMU measurement (for which the actual sensor reading is used), $\mathbf{b} = [\mathbf{b}^{g^\mathsf{T}} \;\; \mathbf{b}^{a^\mathsf{T}}]^\mathsf{T}$ is the IMU bias, and $\mathbf{w}^{\mathrm{IMU}} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{\mathrm{IMU}})$ is the noise associated with the IMU. One can easily obtain the Jacobian of the IMU kinematics with respect to the bias by realizing that the bias is a simple subtractive quantity to the input, and as such, perturbing only the bias,

$$\delta\boldsymbol{\xi}_k = -\mathbf{L}_{k-1}\delta\mathbf{b}_{k-1} \qquad \text{(right)}, \tag{34}$$

$$\delta\boldsymbol{\xi}_k = -\mathbf{Ad}(\bar{\mathbf{T}}_k)\mathbf{L}_{k-1}\delta\mathbf{b}_{k-1} \qquad \text{(left)}. \tag{35}$$

The bias is often jointly estimated with the extended pose, in which case the full problem state becomes $\mathbf{x} = (\mathbf{T}, \mathbf{b}) \in SE_2(3) \times \mathbb{R}^6$. The bias follows a random walk motion model, which is exactly discretized over a small interval with

$$\mathbf{b}_k = \mathbf{b}_{k-1} + \Delta t \mathbf{w}_{k-1}^{\mathrm{bias}}, \tag{36}$$

where $\mathbf{w}^{\text{bias}} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{\text{bias}})$. Defining $\delta\mathbf{x} = [\delta\boldsymbol{\xi}^\mathsf{T} \ \ \delta\mathbf{b}^\mathsf{T}]^\mathsf{T}$, everything up until now can be summarized with the full linearized discrete-time process model:

$$\begin{bmatrix} \delta\boldsymbol{\xi}_k \\ \delta\mathbf{b}_k \end{bmatrix} = \begin{bmatrix} \mathbf{Ad}(\mathbf{U}_{k-1}^{-1}) & -\mathbf{L}_{k-1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\xi}_{k-1} \\ \delta\mathbf{b}_{k-1} \end{bmatrix} + \begin{bmatrix} \mathbf{L}_{k-1} & \mathbf{0} \\ \mathbf{0} & \Delta t \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{k-1}^{\text{IMU}} \\ \mathbf{w}_{k-1}^{\text{bias}} \end{bmatrix} \quad \text{(right)} \tag{37}$$

$$\begin{bmatrix} \delta\boldsymbol{\xi}_k \\ \delta\mathbf{b}_k \end{bmatrix} = \begin{bmatrix} \mathbf{Ad}(\mathbf{G}_{k-1}) & -\mathbf{Ad}(\bar{\mathbf{T}}_k)\mathbf{L}_{k-1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\xi}_{k-1} \\ \delta\mathbf{b}_{k-1} \end{bmatrix} + \begin{bmatrix} \mathbf{Ad}(\bar{\mathbf{T}}_k)\mathbf{L}_{k-1} & \mathbf{0} \\ \mathbf{0} & \Delta t \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{k-1}^{\text{IMU}} \\ \mathbf{w}_{k-1}^{\text{bias}} \end{bmatrix} \quad \text{(left).} \tag{38}$$

4 Preintegration

The compact form in (16) makes preintegration trivial, as two poses related at arbitrary times $k = i$ and $k = j$ can be related by

$$\mathbf{T}_j = \left(\prod_{k=i}^{j-1} \mathbf{G}_k\right) \mathbf{T}_i \left(\prod_{k=i}^{j-1} \mathbf{U}_k\right), \tag{39}$$

$$= \Delta\mathbf{G}_{ij}\mathbf{T}_i\Delta\mathbf{U}_{ij} \tag{40}$$

which is obtained through direct iteration. Equation (40) will be referred to as the *preintegrated process model*.The term $\Delta\mathbf{G}_{ij}$ is constant and does not depend on the state or measurements. The term $\Delta\mathbf{U}_{ij}$ can be considered the "relative motion increment" (RMI) generated from the IMU measurements.

4.1 Noise propagation

An uncertainty associated with this RMI can be defined such that

$$\Delta\mathbf{U}_{ij} = \Delta\bar{\mathbf{U}}_{ij}\text{Exp}(\mathbf{w}_{ij}), \qquad \text{where} \qquad \Delta\bar{\mathbf{U}}_{ij} = \prod_{k=i}^{j-1} \boldsymbol{\Delta}_k \text{Exp}(\bar{\boldsymbol{\upsilon}}), \quad \mathbf{w}_{ij} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{ij}), \tag{41}$$

where $\boldsymbol{\upsilon}$ is defined in (25). The only remaining task is to determine the value of $\mathbf{Q}_{ij}$ from the noise statistics at each individual time step. As an intermediate step, consider a small perturbation to the IMU input $\delta\mathbf{u}$. The RMI can be expanded as

$$\Delta\mathbf{U}_{ij} \approx \prod_{k=i}^{j-1} \boldsymbol{\Delta}_k \text{Exp}(\bar{\boldsymbol{\upsilon}})\text{Exp}(\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{u}_k) \tag{42}$$

$$= \left(\prod_{k=i}^{j-3} \boldsymbol{\Delta}_k \text{Exp}(\bar{\boldsymbol{\upsilon}})\text{Exp}(\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{u}_k)\right)$$
$$\times \boldsymbol{\Delta}_{j-2}\text{Exp}(\bar{\boldsymbol{\upsilon}}_{j-2}) \underbrace{\text{Exp}(\boldsymbol{\mathcal{J}}_{j-2}\boldsymbol{\Upsilon}_{j-2}\delta\mathbf{u}_{j-2})\boldsymbol{\Delta}_{j-1}\text{Exp}(\bar{\boldsymbol{\upsilon}}_{j-1})}_{\boldsymbol{\Delta}_{j-1}\text{Exp}(\bar{\boldsymbol{\upsilon}}_{j-1})\text{Exp}(\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{j-1j}^{-1})\boldsymbol{\mathcal{J}}_{j-2}\boldsymbol{\Upsilon}_{j-2}\delta\mathbf{u}_{j-2})} \text{Exp}(\boldsymbol{\mathcal{J}}_{j-1}\boldsymbol{\Upsilon}_{j-1}\delta\mathbf{u}_{j-1}) \tag{43}$$

$$\vdots \tag{44}$$

$$= \Delta\bar{\mathbf{U}}_{ij} \prod_{k=i}^{j-1} \text{Exp}(\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{u}_k) \tag{45}$$

$$= \Delta\bar{\mathbf{U}}_{ij} \text{Exp} \underbrace{\left(\sum_{k=i}^{j-1} \mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{u}_k\right)}_{\mathbf{w}_{ij}}, \tag{46}$$

where $\boldsymbol{\mathcal{J}}_k \triangleq \boldsymbol{\mathcal{J}}_\ell(-\bar{\boldsymbol{\upsilon}}_k)$. Substituting $\delta\mathbf{u}_k = \delta\mathbf{w}_k$ into (46) and expanding gives

$$\mathbf{w}_{ij} \approx \sum_{k=i}^{j-1} \mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{w}_k \tag{47}$$

$$= \sum_{k=i}^{j-2} \mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{w}_k + \underbrace{\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{jj}^{-1})}_{\mathbf{1}}\boldsymbol{\mathcal{J}}_{j-1} \boldsymbol{\Upsilon}_{j-1} \delta\mathbf{w}_{j-1} \tag{48}$$

$$= \underbrace{\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{j-1j}^{-1})}_{\triangleq \mathbf{A}_{j-1}} \underbrace{\sum_{k=i}^{j-2} \mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j-1}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{w}_k}_{\mathbf{w}_{ij-1}} + \underbrace{\boldsymbol{\mathcal{J}}_{j-1} \boldsymbol{\Upsilon}_{j-1}}_{\triangleq \mathbf{L}_{j-1}} \delta\mathbf{w}_{j-1}. \tag{49}$$

This naturally leads to a recursive computation scheme,

$$\mathbf{Q}_{ij} = \mathbf{A}_{j-1}\mathbf{Q}_{ij-1}\mathbf{A}_{j-1}^{\mathsf{T}} + \mathbf{L}_{j-1}\mathbf{Q}_{j-1}\mathbf{L}_{j-1}^{\mathsf{T}}, \tag{50}$$

which is initialized with $\mathbf{Q}_{ii} = \mathbf{0}$.

4.2 Incorporating bias

If the IMU biases $\mathbf{b}$ are being estimated in addition to the extended pose, then the state can be defined as $\mathbf{x}_k = (\mathbf{T}_k, \mathbf{b}_k) \in SE_2(3) \times \mathbb{R}^6$ and the preintegration process must propagate uncertainty stemming from both the IMU measurements and the bias random walk model. The state $\mathbf{x}_k$ has 15 degrees of freedom (9 for the pose and 6 for the bias), and hence a full 15×15 covariance matrix $\mathbf{Q}_{ij}$ must be maintained when incrementing an RMI. The state $\mathbf{x}_k$ can be preintegrated according to

$$(\mathbf{T}_j, \mathbf{b}_j) = (\Delta\mathbf{G}_{ij}\mathbf{T}_i\Delta\mathbf{U}_{ij},\ \ \mathbf{b}_i + \Delta\mathbf{b}_{ij}), \qquad \Delta\mathbf{b}_{ij} \triangleq \sum_{k=i}^{j-1} \Delta t \mathbf{w}_k^{\text{bias}} \tag{51}$$

where $\mathbf{w}_k^{\text{bias}}$ is noise associated with the bias random walk. The term $\Delta\mathbf{b}_{ij}$ can be thought of as a trivial equivalent of an RMI, but for the bias, and has a mean of $\mathbf{0}$. One can therefore define an augmented increment $\Delta\mathbf{x}_{ij} = (\Delta\mathbf{U}_{ij}, \Delta\mathbf{b}_{ij})$ which will also have an uncertainty $\mathbf{w}_{ij} = [\mathbf{w}_{ij}^{\text{pose}^{\mathsf{T}}} \ \ \mathbf{w}_{ij}^{\text{bias}^{\mathsf{T}}}]^{\mathsf{T}} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{ij})$ defined such that

$$\Delta\mathbf{x}_{ij} \triangleq (\Delta\bar{\mathbf{U}}_{ij}\mathrm{Exp}(\mathbf{w}_{ij}^{\text{pose}}),\ \ \Delta\bar{\mathbf{b}}_{ij} + \mathbf{w}_{ij}^{\text{bias}}), \tag{52}$$

and the overall task is now to compute $\mathbf{Q}_{ij}$ as a function of the IMU noise and bias random walk noise. To this end, consider a small bias perturbation $\delta\mathbf{b}_k$ such that. The common assumption of constant IMU bias over the preintegration period, that is $\delta\mathbf{b}_i = \delta\mathbf{b}_{i+1} = \ldots = \delta\mathbf{b}_{j-1}$, will be applied. Substituting $\delta\mathbf{u}_k = \delta\mathbf{w}_k^{\text{IMU}} - \delta\mathbf{b}_i$ into (46) gives

$$\Delta\mathbf{U}_{ij} \approx \Delta\bar{\mathbf{U}}_{ij}\mathrm{Exp}\left(\sum_{k=i}^{j-1} \mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \delta\mathbf{w}_k + \underbrace{\left(\sum_{k=i}^{j-1} -\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k\right)}_{\mathbf{B}_{ij}}\delta\mathbf{b}_i\right), \tag{53}$$

where $\mathbf{B}_{ij}$ has been defined as the bias Jacobian, and it is straightforward to build incrementally as measurements are obtained since

$$\mathbf{B}_{ij} \approx \sum_{k=i}^{j-1} -\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k \tag{54}$$

$$= \sum_{k=i}^{j-2} -\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k + \underbrace{-\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{jj}^{-1})}_{\mathbf{1}}\boldsymbol{\mathcal{J}}_{j-1} \boldsymbol{\Upsilon}_{j-1} \tag{55}$$

$$= \underbrace{\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{j-1j}^{-1})}_{\triangleq \mathbf{A}_{j-1}} \underbrace{\sum_{k=i}^{j-2} -\mathbf{Ad}(\Delta\bar{\mathbf{U}}_{k+1j-1}^{-1})\boldsymbol{\mathcal{J}}_k \boldsymbol{\Upsilon}_k}_{\mathbf{B}_{ij-1}} + \underbrace{-\boldsymbol{\mathcal{J}}_{j-1} \boldsymbol{\Upsilon}_{j-1}}_{-\mathbf{L}_{j-1}}. \tag{56}$$

From (53), it is straightforward to show that

$$\mathbf{w}_{ij}^{\text{pose}} \approx \begin{bmatrix} \mathbf{A}_{j-1} & -\mathbf{L}_{j-1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{ij-1}^{\text{pose}} \\ \mathbf{w}_{ij-1}^{\text{bias}} \end{bmatrix} + \begin{bmatrix} \mathbf{L}_{j-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{j-1}^{\text{IMU}} \\ \mathbf{w}_{j-1}^{\text{bias}} \end{bmatrix}. \tag{57}$$

and furthermore that

$$\mathbf{w}_{ij}^{\text{bias}} = \mathbf{w}_{ij-1}^{\text{bias}} + \begin{bmatrix} \mathbf{0} & \Delta t \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{j-1}^{\text{IMU}} \\ \mathbf{w}_{j-1}^{\text{bias}} \end{bmatrix}. \tag{58}$$

There two results can be stacked to give

$$\mathbf{w}_{ij} \approx \begin{bmatrix} \mathbf{A}_{j-1} & -\mathbf{L}_{j-1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \mathbf{w}^{ij} + \begin{bmatrix} \mathbf{L}_{j-1} & \mathbf{0} \\ \mathbf{0} & \Delta t \end{bmatrix} \begin{bmatrix} \mathbf{w}_{j-1}^{\text{IMU}} \\ \mathbf{w}_{j-1}^{\text{bias}} \end{bmatrix}, \tag{59}$$

which can be used to compute to propagate the full 15×15 covariance matrix of $\mathbf{w}_{ij}$.

4.3 Prediction using RMI

A final step is to actually predict an extended pose and its covariance, given an RMI and its covariance. This is done with the equation (51), which can be perturbed on both the left or right,

$$\bar{\mathbf{T}}_j \operatorname{Exp}(\delta \boldsymbol{\xi}_j) = \Delta \mathbf{G}_{ij} \bar{\mathbf{T}}_i \operatorname{Exp}(\delta \boldsymbol{\xi}_i) \Delta \bar{\mathbf{U}}_{ij} \operatorname{Exp}(\mathbf{w}_{ij}^{\text{pose}} + \mathbf{B}_{ij} \delta \mathbf{b}_i) \qquad \text{(right)}, \tag{60}$$

$$\operatorname{Exp}(\delta \boldsymbol{\xi}_j) \bar{\mathbf{T}}_j = \Delta \mathbf{G}_{ij} \operatorname{Exp}(\delta \boldsymbol{\xi}_i) \bar{\mathbf{T}}_i \Delta \bar{\mathbf{U}}_{ij} \operatorname{Exp}(\mathbf{w}_{ij}^{\text{pose}} + \mathbf{B}_{ij} \delta \mathbf{b}_i) \qquad \text{(left)}, \tag{61}$$

$$\bar{\mathbf{b}}_j + \delta \mathbf{b}_j = \bar{\mathbf{b}}_i + \delta \mathbf{b}_i + \mathbf{w}_{ij}^{\text{bias}}. \tag{62}$$

A sequence of algebraic manipulations and first-order approximations then follows to yield the linearized preintegrated discrete-time dynamics

$$\begin{bmatrix} \delta \boldsymbol{\xi}_j \\ \delta \mathbf{b}_j \end{bmatrix} = \begin{bmatrix} \mathbf{Ad}(\Delta \bar{\mathbf{U}}_{ij}^{-1}) & \mathbf{B}_{ij} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{\xi}_i \\ \delta \mathbf{b}_i \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{j-1}^{\text{pose}} \\ \mathbf{w}_{j-1}^{\text{bias}} \end{bmatrix} \qquad \text{(right)}, \tag{63}$$

$$\begin{bmatrix} \delta \boldsymbol{\xi}_j \\ \delta \mathbf{b}_j \end{bmatrix} = \begin{bmatrix} \mathbf{Ad}(\Delta \mathbf{G}_{ij}) & \mathbf{Ad}(\bar{\mathbf{T}}_j) \mathbf{B}_{ij} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{\xi}_i \\ \delta \mathbf{b}_i \end{bmatrix} + \begin{bmatrix} \mathbf{Ad}(\bar{\mathbf{T}}_j) & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{j-1}^{\text{pose}} \\ \mathbf{w}_{j-1}^{\text{bias}} \end{bmatrix} \qquad \text{(left)}, \tag{64}$$

which are both of the form $\delta \mathbf{x}_j = \mathbf{A}_{ij} \delta \mathbf{x}_i + \mathbf{L}_{ij} \mathbf{w}_{ij}$. The covariance of the preintegrated pose and bias is then given by

$$\mathbf{P}_j = \mathbf{A}_{ij} \mathbf{P}_i \mathbf{A}_{ij}^{\mathsf{T}} + \mathbf{L}_{ij} \mathbf{Q}_{ij} \mathbf{L}_{ij}^{\mathsf{T}}. \tag{65}$$

Appendix

A Solution to linear matrix-valued ODE

Consider the matrix-valued ODE

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}. \tag{66}$$

In the upcoming derivation, the vectorization operator $\mathbf{x} = \mathrm{vec}(\mathbf{X})$ will be used, along with the following identities

$$\begin{aligned}
\mathrm{vec}(\mathbf{ABC}) &= (\mathbf{C}^{\mathsf{T}} \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}), && (67)\\
\mathrm{vec}(\mathbf{AB}) &= (\mathbf{1} \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}) = (\mathbf{B}^{\mathsf{T}} \otimes \mathbf{1})\mathrm{vec}(\mathbf{A}), && (68)\\
\mathbf{A} \oplus \mathbf{B} &= \mathbf{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{B}, && (69)\\
\exp(\mathbf{A} \oplus \mathbf{B}) &= \exp(\mathbf{A}) \otimes \exp(\mathbf{B}). && (70)
\end{aligned}$$

The general solution to the ODE now follows as

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \big((\mathbf{1} \otimes \mathbf{A}) + (\mathbf{B}^{\mathsf{T}} \otimes \mathbf{1})\big)\, \mathbf{x}(t) && (71)\\
&= (\mathbf{B}^{\mathsf{T}} \oplus \mathbf{A})\mathbf{x}(t) && (72)\\
\mathbf{x}(t) &= \exp((\mathbf{B}^{\mathsf{T}} \oplus \mathbf{A})t)\mathbf{x}_0 && (73)\\
&= (\exp(\mathbf{B}^{\mathsf{T}} t) \otimes \exp(\mathbf{A}t))\mathbf{x}_0 && (74)\\
&= \mathrm{vec}\left(\exp(\mathbf{A}t)\mathbf{X}_0 \exp(\mathbf{B}t)\right), && (75)\\
\mathbf{X}(t) &= \exp(\mathbf{A}t)\mathbf{X}_0 \exp(\mathbf{B}t). && (76)
\end{aligned}$$

B Expression for the N matrix

The matrix $\mathbf{N}(\cdot)$ is defined as

$$\begin{aligned}
\mathbf{N}(\boldsymbol{\phi}) &= 2\sum_{n=0}^{\infty} \frac{1}{(n+2)!}(\boldsymbol{\phi}^{\wedge})^n, && (77)\\
&= 2\left(\frac{1}{2!}\mathbf{1} + \frac{1}{3!}\boldsymbol{\phi}^{\wedge} + \frac{1}{4!}\boldsymbol{\phi}^{\wedge 2} + \frac{1}{5!}\boldsymbol{\phi}^{\wedge 3} + \dots\right). && (78)
\end{aligned}$$

and it can be shown that this is equivalent to

$$\mathbf{N}(\boldsymbol{\phi}) = \mathbf{a}\mathbf{a}^{\mathsf{T}} + 2(\frac{1}{\phi} - \frac{\sin\phi}{\phi^2})\mathbf{a}^{\wedge} + 2\frac{\cos\phi - 1}{\phi^2}\mathbf{a}^{\wedge}\mathbf{a}^{\wedge}, \tag{79}$$

and $\mathbf{N}(\mathbf{0}) = \mathbf{1}$. In [4] it is apparently stated that a first-order approximation to $\mathbf{N}(\boldsymbol{\phi})$ is given by

$$\mathbf{N}(\bar{\boldsymbol{\phi}} + \delta\boldsymbol{\phi}) = \mathbf{N}(\bar{\boldsymbol{\phi}}) + \frac{1}{3}\delta\boldsymbol{\phi}^{\wedge}. \tag{80}$$

C Adjoint and inverse expressions for $\mathbf{G}_k, \mathbf{U}_k$ matrices

The "increment matrices" $\mathbf{G}_k, \mathbf{U}_k$ are both of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{C} & \mathbf{a} & \mathbf{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in IE(3) \tag{81}$$

where $\mathbf{C} \in SO(3)$, and form a group under matrix multiplication. which is a group that will arbitrarily be called the *Incremental Euclidean group of dimension* n, $IE(n)$. The inverse of an element of $IE(n)$ is given by

$$\mathbf{X}^{-1} = \begin{bmatrix} \mathbf{C}^\mathsf{T} & -\mathbf{C}^\mathsf{T}\mathbf{a} & \mathbf{C}^\mathsf{T}(c\mathbf{a} - \mathbf{b}) \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}. \tag{82}$$

The adjoint matrix of an arbitrary element of this group is defined in this document to be the unique matrix $\mathbf{Ad}(\mathbf{X})$ such that

$$(\mathbf{Ad}(\mathbf{X})\boldsymbol{\xi})^\wedge = \mathbf{X}\boldsymbol{\xi}^\wedge\mathbf{X}^{-1} \tag{83}$$

where $\boldsymbol{\xi}^\wedge$ is the belongs to the Lie algebra of $SE_2(3)$. Note that there is a subtle difference from the usual definition of the adjoint matrix, which would require that $\boldsymbol{\xi}^\wedge$ belong to the Lie algebra of $IE(3)$. Nevertheless, $\mathbf{Ad}(\mathbf{X})$ is given by

$$\mathbf{Ad}(\mathbf{X}) = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{a}^\wedge\mathbf{C} & \mathbf{C} & \mathbf{0} \\ -(c\mathbf{a} - \mathbf{b})^\wedge\mathbf{C} & -c\mathbf{C} & \mathbf{C} \end{bmatrix}. \tag{84}$$

It is straightforward to show that the increment matrices $\mathbf{G}_k, \mathbf{U}_k$ therefore have the following inverses.

$$\mathbf{G}_k^{-1} = \begin{bmatrix} \mathbf{1} & -\Delta t\mathbf{g} & -(\Delta t^2/2)\mathbf{g} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}, \tag{85}$$

$$\mathbf{U}_k^{-1} = \begin{bmatrix} \boldsymbol{\Omega}^\mathsf{T} & -\Delta t\boldsymbol{\Omega}^\mathsf{T}\mathbf{J}_\ell\mathbf{a} & \Delta t^2\boldsymbol{\Omega}^\mathsf{T}(\mathbf{J}_\ell - \mathbf{N}(\Delta t\boldsymbol{\omega})\mathbf{a}) \\ 0 & 1 & -\Delta t \\ 0 & 0 & 1 \end{bmatrix} \tag{86}$$

where $\boldsymbol{\Omega} = \exp(\Delta t\boldsymbol{\omega}^\wedge)$.

References

[1] C. Forster, L. Carlone, F. Dellaert, and D. Scaramuzza, "On-Manifold Preintegration for Real-Time Visual-Inertial Odometry," *IEEE Trans. Robotics*, vol. 33, no. 1, pp. 1–21, 2017.

[2] A. Barrau and S. Bonnabel, "A Mathematical Framework for IMU Error Propagation with Applications to Preintegration," in *International Conference on Robotics and Automation (ICRA)*, 2020, pp. 5732–5738.

[3] M. Brossard, A. Barrau, P. Chauchat, and S. Bonnabel, "Associating Uncertainty to Extended Poses for on Lie Group IMU Preintegration with Rotating Earth," pp. 1–16, 2020. arXiv: `2007.14097`. [Online]. Available: `http://arxiv.org/abs/2007.14097`.

[4] T. D. Barfoot, *State Estimation for Robotics - 2nd Edition*, 2nd ed. 2022.